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On the EXTRACTION *of the* CUBIC *and other* ROOTS.

By ——— ———.

Communicated by the Rev. M. YOUNG, *D. D.*

Read January 1, 1787.

WHEN the cubic root of any number is required true to many places of decimals, let the given number be reſolved into a binomial, making the firſt term a cubic number, and the ſum or difference of the terms equal to the given number. Let both terms be then divided by the firſt, which makes the firſt term 1, and the ſecond a fraction; and let the cubic root of this reſulting binomial be reſolved into an infinite ſeries by the binomial theorem; and let as many terms of it be calculated, as give it true to two or three more than the required number of places; and laſtly, let this root be multiplied into the cubic root of the term which was made the diviſor above, and this gives the number required.

Thus:—Let it be required to find the cubic root of 28, true to ten places of decimals. Firſt make $28=27+1$, and then dividing by 27, a new binomial $1+\frac{1}{27}$ is had, whoſe cubic root is to be reſolved into an infinite ſeries. Now any power of it whatever may be found by the following ſeries,

$$1 . \frac{m}{1.27} . \frac{m-1}{2.27} . \frac{m-2}{3.27} . \frac{m-3}{4.27} . \frac{m-4}{5.27} . \frac{m-5}{6.27}, \text{ \&c.}$$

where m is the index of the power. In the preſent caſe $m=\frac{1}{3}$, and then the above ſeries reduced into numbers will ſtand thus:

$$1 . \frac{+1}{3\cdot 27} . \frac{-1}{3\cdot 27} . \frac{-5}{9\cdot 27} . \frac{-2}{3\cdot 27} . \frac{-11}{15.27} . \frac{-7}{9\cdot 27} . \frac{-17}{21.27}, \text{ \&c.}$$

And it is plain that of the terms of the infinite ſeries, the firſt and ſecond will be affirmative, the third negative, and the following ones alternately affirmative and negative: and the work of calculating the terms will ſtand thus:

$+1,000\,000$ 1ſt term.
This multiplied by $\frac{+1}{3\cdot 27}$, gives $+0,012\,345\,679\,0123$. 2d term.
Which multiplied by $\frac{-1}{3\cdot 27}$, gives $-,000\,152\,415\,7902$. 3d term.
And 3d term multiplied by $\frac{-5}{9\cdot 27}$, gives - $+3\,136\,1273$. 4th term.
And 4th term multiplied by $\frac{-2}{3\cdot 27}$, gives - - $-77\,4352$. 5th term.
5th term multiplied by $\frac{-11}{15.27}$, gives - - $+2\,1031$. 6th term.
6th term multiplied by $\frac{-7}{9\cdot 27}$, gives - - - -605. 7th term.
And 7th term multiplied by $\frac{-17}{21.17}$, gives - - - $+18$. 8th term.

HAVING calculated the above eight terms, and ſeeing that the whole value of a ninth term would not amount to unity in the laſt place of decimals, that and all the following terms may be neglected; and the ſum of theſe eight, added together according to their ſigns, as follows, is the cubic root of the binomial $1+\frac{1}{27}$:

1ſt term	+1,000 000 000 0000
2d term	+0,012 345 679 0123
4th term	+3 136 1273
6th term	+2 1031
8th term	+18
Sum of affirmative terms	+1,012 348 817 2445
3d term	—152 415 7902
5th term	—77 4352
7th term	—605
Sum of negative terms	—152 493 2859
Cubic root of $1+\frac{1}{27}$	1,012 196 323 9586

AND ſince $27+1$ $(=28) : 1+\frac{1}{27} :: 27 : 1$; $\sqrt[3]{28} : \sqrt[3]{1+\frac{1}{27}} :: 3 : 1$. If therefore the cubic root of $1+\frac{1}{27}$ be multiplied into 3, the product will be the cubic root of 28, which is therefore 3,0365889718758.

But ſince each of theſe eight terms calculated above is imperfect, being leſs than the truth, the ſum of them may be different from the truth. Two or three of the laſt figures muſt be omitted, and then we can be ſure that the reſt are true*.

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* What figures are to be omitted, in order that the remaining decimal may be true, is to be determined by finding out the greater and leſs limits of the decimal, and as far as theſe agree, ſo far we may be certain that the decimal is accurate. Now as a decimal firſt ariſes, it is leſs than the truth, becauſe certain figures are omitted; this therefore we may conſider as the leſs limit; and if to the laſt figure we add an unit, we ſhall have the greater limit. But in the addition, multiplication, ſubtraction and diviſion of decimals, the limits of the ſum, product, difference, and quotient may be determined by the following rules:

1. In addition, the limits of the ſum are had by adding together the greater limits of the quantities to be added, for the greater limit of the ſum; and the leſs limits together for the leſs limit of the ſum.

Thus, if the greater and leſs limits of one of the quantities be 123,5 and 123,2; and the greater and leſs limits of the other be 43,2 and 43,1; then will 166,7 and 166,3 be the greater and leſs limits of the ſum.

2. In multiplication, the limits of the product are had by multiplying together the greater limits of the factors, for the greater limit of the product; and the leſs limits together, for the leſs limit of the product.

Thus, if the limits of one quantity be 10,502 and 10,5, and the limits of the other be 3,216 and 3,215; then the limits of the product will be 33,774432 and 33,7575.

3. In ſubtraction, the greater limit of the leſs quantity is to be ſubtracted from the leſs limit of the greater, for the leſs limit of the difference; and the leſs limit of the leſs from the greater limit of the greater, for the greater limit of the difference.

If the given number had been 26, it muſt then have been reſolved into 27—1, and dividing by 27, the new binomial would have been $1 - \frac{1}{27}$, and the terms of the ſeries $1 . \frac{m}{3 \cdot 27} . \frac{m-1}{3 \cdot 27}$, &c. when reduced to numbers would have been $1 . \frac{-1}{3 \cdot 27} . \frac{+1}{3 \cdot 27} . \frac{+5}{9 \cdot 27} . \frac{+2}{3 \cdot 27} . \frac{+11}{15 \cdot 27} . \frac{+7}{9 \cdot 27} . \frac{+17}{21 \cdot 27}$, &c. And of the terms of the infinite ſeries reſulting, the firſt would have been affirmative, and all the reſt negative, which is always the caſe when the ſecond term of the binomial is negative, as in the following example :

Thus, if the limits of the greater quantity be 166,7 and 166,3, and the limits of the leſs quantity be 43,2 and 43,1, then will 123,6 and 123,1 be the limits of their difference.

4. In diviſion, the greater limit of the dividend is to be divided by the leſs limit of the diviſor, for the greater limit of the quotient; and the leſs limit of the dividend by the greater limit of the diviſor, for the leſs limit of the quotient.

Thus, if the limits of the dividend be 33,774432 and 33,7575, and the limits of the diviſor be 3,216 and 3,215, then will 10,506 and 10,496 be the limits of the quotient.

Vide Mirifici Logarith. Canonis Conſtructio. Edinburg 1619; *vel Lugduni* 1620.

In the example given above 1,0123488172445 is the leſs limit of the ſum of the affirmative terms, and 1,0123488172449 the greater limit. In like manner, ,0001524932859 is the leſs limit of the ſum of the negative terms, and ,0001524932862 the greater limit ; therefore according to rule 3. 1,0121963239583, and 1,0121963239590, are the leſs and greater limits of the ſum of all theſe terms added together according to their ſigns. And conſequently 3,0365889718749 and 3,0365889718770 are the leſs and greater limits of the cubic root of the number 28.

LET it be required to find the cubic root of 210, true to twelve places of decimals. The neareſt cube number to 210 is 216, the cube of 6. Therefore making the binomial 216—6, and dividing by 216, the new binomial is $1-\frac{6}{216}=1-\frac{1}{36}$, and the ſeries which yields any power of it, is $1 . \frac{m.-1}{1\cdot 36} . \frac{m-1.-1}{2\cdot 36} . \frac{m-2.-1}{3\cdot 36} . \frac{m-3.-1}{4\cdot 36}$, &c. Which reduced to numbers is $1 . \frac{-1}{3\cdot 36} . \frac{+1}{3\cdot 36} . \frac{+5}{9\cdot 36} . \frac{+1}{9\cdot 6} . \frac{+11}{15\cdot 36} . \frac{+7}{9\cdot 36} . \frac{+17}{21\cdot 36} . \frac{+5}{6\cdot 36}$, &c. and the calculation ſtands thus:

	1,000 000 000 000 000.	1ſt term.
1ſt term multiplied by $\frac{-1}{3\cdot 36}$	—9 259 259 259 259.	2d term.
2d term multiplied by $\frac{+1}{3\cdot 36}$	—85 733 882 030.	3d term.
3d term multiplied by $\frac{+5}{9\cdot 36}$	—1 323 053 735.	4th term.
4th term multiplied by $\frac{+1}{6\cdot 9}$	—24 500 995.	5th term.
5th term multiplied by $\frac{+11}{15\cdot 36}$	—499 094.	6th term.
6th term multiplied by $\frac{+7}{9\cdot 36}$	—10 782.	7th term.
7th term multiplied by $\frac{+17}{21\cdot 36}$	—242.	8th term.
8th term multiplied by $\frac{+5}{6\cdot 36}$	—5.	9th term.
Sum of the negative terms	—0,009 346 341 206 142	
Sum of all the terms, added according to their ſigns	+0,990 653 658 793 858	
The ſum of all the terms multiplied by 6	5,943 921 952 763 148	

HAVING found theſe nine terms, and neglecting all the reſt for the reaſon given above, let them be added according to their ſigns, and their ſum is the cubic root of the binomial $1-\frac{1}{36}$. And ſince $216-6\ (=210) : 1-\frac{1}{36} :: 216 : 1$, it will be $\sqrt[3]{210} : \sqrt[3]{1-\frac{1}{36}} :: 6 : 1$. Therefore let the cubic root of $1-\frac{1}{36}$, found above, be multiplied into 6, and the number reſulting, which (neglecting the three laſt figures *) is 5,943921952763, is the cubic root of 210, true to twelve places of decimals, as was required.

THE given numbers in theſe two examples, were purpoſely choſen ſuch as ſhould make the operation eaſy. But in other examples, the difference between the given number and the cube number next greater or next leſs, may be ſo great in proportion to that cube, as to make the fraction, (the ſecond member of the binomial,) too large: In theſe caſes the ſeries will converge ſo ſlowly, that the labour will be almoſt intolerable. Thus, if the given number were 13, it muſt be made equal either to $8+5$, or to $27-14$, and the binomial will be either $1+\frac{5}{8}$, or $1-\frac{14}{27}$. If the given number be 3, the binomial will be $1-\frac{5}{8}$; and if the given number be 2, the binomial will be $1-\frac{3}{4}$. In all theſe caſes, and ſuch like, the ſeries

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* As before, the numbers 0,009346341206142 and 0,009346341206150 are the leſs and greater limits of the ſum of the negative terms; and the numbers 0,990653658793858 and 0,990653658793850 are the greater and leſs limits of the ſum of all the terms added together according to their ſigns. Conſequently 5,943921952763148 and 5,943921952763100 are the greater and leſs limits of the cubic root of the number 210, which agree even to thirteen places of decimals; and therefore the root itſelf is ſo far accurately calculated.

will converge ſo ſlowly, that a very great number of terms muſt be calculated, and therefore the labour will be immenſe. This, however, is an inconvenience that may be remedied; for inſtead of reſolving the given number into a binomial, let it be multiplied into ſome cube number, ſo as that the product may be nearly equal to ſome other cube number; and let this product be reſolved into a binomial, and its root found; which being multiplied into the cubic root of the diviſor, as above, and divided by the cubic root of the number into which the given number was multiplied, the quotient is the root required.

THUS if the cubic root of 2 be required, it might be reſolved into 8—6, and dividing by 8 it would become $1-\frac{3}{4}$. But this is to be rejected for the reaſon given above. Multiplying therefore 2 into 8, the product 16 gives the binomial $1-\frac{11}{27}$. But here alſo the fraction $\frac{11}{27}$, (though leſs than the former,) is too great. Multiplying then 2 into 27, the next cube number, the product 54 gives the binomial $1-\frac{10}{64}$, where the fraction is ſtill leſs, and might be uſed, only that on multiplying 2 into 64, the next cube number, the product 128 gives the binomial $1+\frac{3}{125}$, which is as convenient as can be deſired.

FIND therefore its cubic root, by the method above, and multiply it into 5, (the cubic root of the denominator of the fraction,) and the product is the root of 128: And this root being divided by 4, (the cubic root of the multiplier 64,) the quotient is the cubic root of 2, as was deſired.

THAT this method may always be uſed when there is occaſion, it is neceſſary to ſolve the following

PROBLEM.

A NUMBER, not a cube number, being given, it is required to find a cube number, which multiplied into the given number, ſhall give a product nearly equal to ſome other cube number.

SUPPOSE n a given number, not a cube, it is required to find two other numbers, a the greater, and b the leſs, ſo as that a^3 ſhall be nearly equal to nb^3, or that the fraction $\frac{a^3-nb^3}{a^3}$, or $\frac{nb^3-a^3}{a^3}$ ſhall be very ſmall.

WHEN the given number is ſmall, the numbers a and b may be found by a few and eaſy trials; but if it be great, the ſolution by trials is very difficult. But a general and direct ſolution of it was ſuggeſted by the ſolution of a problem, nearly of the ſame nature, by Doctor Wallis, and publiſhed by him in the *Commercium Epiſtolicum*, at the end of the ſecond volume of his works. An example will ſufficiently ſhew the method uſed in the ſolution.

LET the given number be 13; and a^3 will be nearly equal to $13b^3$. Call the difference y, and we ſhall have this equation, $a^3=13b^3\pm y$. We are next to find the limits of a with reſpect to b; that is, to find the multiples of b which are next leſs and next greater than a. And ſince the cubic root of

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13 is between 2 and 3, $2b$ and $3b$ are the limits, of which $2b$ is nearer to a; and being leſs than a, make $a=2b+c$, and by ſubſtituting we have this equation:

$$8b^3+12b^2c+6bc^2+c^3=13b^3\pm y;$$

which by reduction becomes

$$5b^3=12b^2c+6bc^2+c^3\mp y.$$ *Firſt anſwer.*

In this equation alſo we are to determine the limits of b the greater quantity with reſpect to c the leſs; which may be done by ſubſtituting ſome multiple of c for b: and according as the left ſide of the reſulting equation is leſs or greater than the right ſide, that multiple is leſs or greater than the truth. And to avoid unneceſſary trials, let the coefficient of the firſt term on the right ſide of the equation, be divided by the coefficient of the term that ſtands on the left ſide; and the quotient, (neglecting the fraction,) multiplied into the leſs quantity, will be one of the limits, or near it. And when all the terms on the right ſide have affirmative ſigns, the limit thus found will be leſs than the truth; but when ſome of them have negative ſigns, it will probably be greater than the truth. Thus, in the laſt equation above, divide 12 by 5, and the quotient being 2, make $b=2c$, and upon trying it will be found, that $2c$ is leſs than the truth; therefore make $b=3c$, and upon trial $3c$ will be found greater than the truth. Therefore $2c$ and $3c$ are the limits of b, of which $3c$ ſeems nearer to the truth; therefore make $b=3c-d$, and by ſubſtituting, this equation reſults,

$$135c^3-135c^2d+45cd^2-5d^3=108c^3-72c^2d+12cd^2+18c^3-6c^2d+c^3\mp y;$$

and by reduction,

$$8c^3=57c^2d-33cd^2+5d^3\mp y.$$ *Second anſwer.*

HERE alſo, as before, we are to find the limits of c the greater quantity with reſpect to d the leſs; and they appear to be $6d$ and $7d$; of which $6d$ ſeems the more convenient. Making therefore $c=6d+e$, there reſults this equation,

$$1728d^3+864d^2e+144de^2+8e^3=2052d^3+684d^2e+57de^2-198d^3-33d^2e+5d^3\mp y;$$

which by reduction becomes

$$131d^3=213d^2e+87de^2+8e^3\pm y.\ \textit{Third anſwer.}$$

HERE alſo the limits of d being found to be e and $2e$; and $2e$ being nearer the truth, make $d=2e-f$, and this equation reſults,

$$1048e^3-1572e^2f+786ef^2-131f^3=852e^3-852e^2f+213ef^2+174e^3-87e^2f+8e^3\pm y;$$

which by reduction becomes

$$14e^3=633e^2f-573ef^2+131f^3\pm y.\ \textit{Fourth anſwer.}$$

HERE the limits of e are $44f$ and $45f$. Therefore making $e=44f+g$, this equation reſults,

$$\begin{aligned}1192576f^3+81312f^2g+1848fg^2+14g^3=1225488f^3&+55704f^2g+633fg^2\pm y\\ -25212f^3&-573f^2g\\ +131f^3&\end{aligned}$$

which by reduction becomes

$$7831f^3=26181f^2g+1215fg^2+14g^3\mp y.\ \textit{Fifth anſwer.}$$

AND the limits of f being $3g$ and $4g$, make $f=3g+h$, and this equation reſults,

$$\begin{aligned}211437g^3+211437g^2h+70479gh^2+7831h^3=235629g^3&+157086g^2h+26181gh^2\mp y\\ +3645g^3&+1215g^2h\\ +14g^3&\end{aligned}$$

and by reduction

$$27851g^3=53136g^2h+44298gh^2+7831h^3\pm y.\ \textit{Sixth anſwer.}$$

In the ſame manner, may be found as many anſwers as any one ſhall pleaſe, and in every ſucceeding anſwer the denominator of the fraction, (the ſecond member of the binomial,) becomes greater; but the fraction itſelf, (which is ultimately ſought by this inveſtigation,) becomes leſs. Now by the help of any of the above equations marked *firſt*, *ſecond*, *third*, &c. *anſwers*, may the values of a and b be found. Thus, in any of thoſe equations make the leſs quantity $=0$ and the greater $=1$, and from theſe, by going backward, may ſucceſſively be found the values of all the letters that had been thrown off by the ſeveral ſubſtitutions, until we come to b and a; and the value of y will be the coefficient of the term on the left ſide of that equation, where the greater and leſs quantities were made $=1$ and 0. Thus, in the fourth anſwer, if we make $f=0$ and $e=1$, then will

$$d=(2e-f=)\ 2$$
$$c=(6d+e=)\ 13$$
$$b=(3c-d=)\ 37$$
$$a=(2b+c=)\ 87$$

Now the cube of 37 is 50653, which multiplied by 13 gives 658489, and the cube of 87 is 658503; and the difference is 14, the coefficient of the term on the left ſide of that equation: and when the ſign of y is $\pm$, as in the firſt equation, a^3 will be greater than the multiple of b^3, and therefore in the binomial, the ſign of the fractional part will be negative: But when the ſign of y is $\mp$, a^3 will be leſs, and the fractional part of the binomial will have a poſitive ſign.

In the following table are put down the values of b, a, and y, according to each of the ſix anſwers inveſtigated above; and annexed are the binomials reſulting from theſe anſwers: any one of whoſe roots, being reſolved into an infinite ſeries, will ſerve for finding the cubic root of 13.

	Value of b.	Value of a.	Value of y.	The reſulting binomial, whoſe root is to be reſolved into an infinite ſeries.
By 1ſt anſwer	1	2	5	$1+\frac{5}{8}$
2d	3	7	8	$1+\frac{8}{343}$
3d	17	40	131	$1-\frac{131}{64000}$
4th	37	87	14	$1-\frac{14}{658503}$
5th	1611	3788	7831	$1+\frac{7831}{54353799872}$
6th	4870	11451	27851	$1-\frac{27851}{150151696685 1}$

If any one ſhall ever uſe this method of extracting the cubic root, it muſt be left to his judgment, to determine how far this inveſtigation is to be continued. On the one hand the fraction, (the ſecond member of the binomial to be evolved,) becomes leſs, and therefore the ſeries will converge the faſter. But, on the other hand, the trouble of continuing the inveſtigation, when the numbers become large, and of dividing by a large diviſor afterwards, is much increaſed. Indeed when the given number conſiſts of many places of figures, the labour of dividing by a large diviſor cannot be avoided. But the trouble of inveſtigation may be ſaved, as the cube number next greater or next leſs than the given number, may be made the firſt member of the binomial, into which the given number is reſolved.

If the given number be a decimal fraction, or an integer with a decimal annexed, it will be convenient to reduce it to an integral number, by removing the *nota ſeparatrix* to the right hand over a number of places which muſt always be diviſible by 3: (one or two cyphers being added after the ſignificant figures, when neceſſary, to make the number of decimal places a multiple of 3:) And when the root of the integral number is found, as many of its integral figures are now to be added to the decimals, as there were ternaries of decimal figures, before added to the integers.

If the given number be a vulgar fraction, (either proper or improper,) let the ſquare of the denominator be multiplied by the numerator, and the cubic root of the product be found as above; and let the given denominator be ſubſcribed under this root, if a vulgar fraction be ſufficient: Or let this root be divided by it, if a decimal fraction be neceſſary.

By a like proceſs *(mutatis mutandis)* any root may be extracted out of a given number: but when the index of the root is any term of the duple progreſſion, beginning from unity, the operation, as is well known, may be otherwiſe performed in a more ſimple manner.